

# 'Convincing and Proving' Tasks

## Malcolm Swan

Mathematics Education  
University of Nottingham  
Malcolm.Swan@nottingham.ac.uk

## Jim Ridgway

School of Education  
University of Durham  
Jim.Ridgway@durham.ac.uk

### WHY USE CONVINCING AND PROVING TASKS?

Proof lies at the heart of mathematics and mathematical thinking, yet many students will have had little exposure to proofs in high school. This CAT introduces the notions of convincing and proving and illustrates several kinds of proof commonly encountered in mathematics. These tasks are intended to assess how well students are able to argue logically, use examples and counterexamples to support their reasoning and identify breakdowns in rational argument. In addition, some tasks reveal common student misconceptions students make in their reasoning.

### WHAT ARE CONVINCING AND PROVING TASKS?

These tasks are of two types.

- The first type asks students to evaluate a set of statements as "always, sometimes or never true". Students are expected to offer examples, counterexamples, and reasons for their decisions.
- The second type requires the student to evaluate "proofs" and distinguish the correct from the flawed.

### WHAT IS INVOLVED?

<b>Instructor Preparation Time:</b>	Minimal if using existing tasks.
<b>Preparing Your Students:</b>	Students will need some coaching on their first task.
<b>Class Time:</b>	45 minutes.
<b>Disciplines:</b>	Appropriate for students who are expected to be able to argue logically. Requires some basic algebraic skills.
<b>Class Size:</b>	Any.
<b>Special Classroom/Technical Requirements:</b>	None.
<b>Individual or Group Involvement:</b>	Either.
<b>Analyzing Results:</b>	Intensive for formal scoring for large classes. Best used as an informal way to get your students thinking mathematically.
<b>Other Things to Consider:</b>	Some of the later tasks are intellectually demanding .

## Description

In the first collection of tasks, students are asked to evaluate a series of statements. These typically concern mathematical results or hypotheses, such as "The square of a number is greater than the number". Students are invited to classify each statement as "always true," "sometimes true," or "never true" and offer reasons for their decision. (In this case, for example, they should decide that the statement is sometimes true - when the number is negative or greater than one.) The best responses will contain convincing explanations and proofs; the weaker responses will typically contain just a few examples and counter-examples. These tasks vary in difficulty, according to the statements being considered and the difficulty of providing a convincing or rigorous explanation. These tasks can also diagnose student misconceptions, which often arise from over-generalizing from limited domains.

In the second collection of tasks, students are asked to evaluate "proofs," some of which are correct and others which are flawed. (For example, in one question, three 'proofs' of the Pythagorean theorem are given). The flawed "proofs" may be:

- inductive rather than deductive arguments which only work with special cases;
- arguments which assume the result to be proved; or
- arguments which contain invalid assumptions.

There are also some partially correct proofs that contain large unjustified jumps in reasoning. In these tasks, students are expected to identify the most convincing proof and provide critiques for the remaining attempts.

Examples of the two types of tasks appear below:

### 1. "Always, Sometimes or Never True"

The aim of this assessment is to provide the opportunity for you to:

- test statements to see how far they are true;
- provide examples or counterexamples to support your conclusions; and
- provide convincing arguments or proofs to support your conclusions.

For each statement, say whether it is always, sometimes or never true. You must provide **several examples or counterexamples** to support your decision. Try also to provide convincing reasons for your decision. You may even be able to provide a **proof** in some cases.

**The more digits a number has, then the larger is its value.**

Is this always, sometimes or never true? .....

Reasons or examples:

### 2. Critiquing 'Proofs'

The aim of this assessment is to provide the opportunity for you to:

- evaluate 'proofs' of given statements and identify which are correct; and
- identify errors in 'proofs.'

1. Consecutive Addends

Here are three attempts at proving the following statement:

When you add three consecutive numbers, your answer is always a multiple of three.

Look carefully at each attempt. Which is the best 'proof'? Explain your reasoning as fully as possible.

*Attempt 1:*

$1 + 2 + 3 = 6$	$3 \times 2 = 6$
$2 + 3 + 4 = 9$	$3 \times 3 = 9$
$3 + 4 + 5 = 12$	$3 \times 4 = 12$
$4 + 5 + 6 = 15$	$3 \times 5 = 15$
$5 + 6 + 7 = 18$	$3 \times 6 = 18$

And so on.  
So it must be true.

*Attempt 2:*

$3 + 4 + 5$   
The two outside numbers (3 and 5) add up to give twice the middle number (4).  
So all three numbers add to give three times the middle number.  
So it must be true

*Attempt 3:*

Let the numbers be:  
 $n, n + 1$  and  $n + 2$   
Since  
 $n + n + 1 + n + 2 = 3n + 3 = 3(n + 1)$   
It is clearly true.

The best proof is attempt number .....

This is because .....

### **Assessment Purposes**

The purposes underlying these tasks are twofold:

- To probe the conceptual understanding of students concerning the given statements
- To assess how well students are able to justify and prove.

### **Limitations**

'Convincing and Proving' tasks are rather pure and mathematical in flavor. They have been constructed in this manner so that the situations are well-defined and unambiguous. Some students may not like this esoteric approach. They also require the use and analysis of algebra.

### **Teaching Goals**

- Students learn that one cannot 'prove' a result by simply generating illustrative examples
- Students learn that a single counter-example can sink a whole 'proof'
- Students learn that results which are clearly true in limited domains are not necessarily true in wider domains.
- Students learn to analyze arguments and identify flaws in them.
- Students develop an appreciation of the nature of proof.
- Students learn strategies for testing proofs.
- Students learn to debug some common errors in proofs.

### Suggestions for Use

#### *Introducing 'Convincing and Proving' tasks for the first time*

Many students will find these tasks unfamiliar since many have been required to provide proofs only for well-known results in geometry (which are often produced on demand after rote learning). Students are rarely asked to consider statements and evaluate their validity.

The first time you use these tasks, we suggest that you offer just one statement and allow students to discuss whether it is always, sometimes or never true, in small groups. For example, you can use the statement:

*When you add two numbers, you get the same result as when you multiply them.*

At first glance, most students would probably say that such a statement is 'false' or only true when the numbers are both zero or both two. It is worth pausing at this stage and asking whether there are any more cases. If no suggestions are forthcoming, or students appear quite confident that there aren't any more, suggest they try  $\frac{3}{2}$  and 3. They are often surprised by this example. Now ask them to try to find *every* case where the statement is true.

After a while, you might like to display the following pairs of numbers that result in a true statement and ask students for conjectures on a general rule.

$$\frac{2}{1}, 2$$

$$\frac{3}{2}, 3$$

...

Some students may suggest continuing the pattern and trying

$$\frac{4}{3}, 4$$

or even

$$\frac{n}{n-1}, n$$

They may like to try checking other special cases (possibly containing decimals or negative numbers) or using algebra. From this, you can ask the students if they feel they have a proof that this set of answers is the set of all answers that work. If it is not, then ask the students what to do and allow them time to discuss this issue. They should realize that there may be many more answers which they haven't yet thought about.

Finally a 'proof' may be established:

### **This statement is sometimes true.**

Proof:

Suppose one number is  $x$  and one number is  $y$ .

The statement says that:  $x + y = xy$

This is true if and only if:  $y = x/(x - 1)$

So the statement works for any pair of numbers:  $(x, x/(x - 1))$

It does not work if this condition does not hold.

Try to get students to explain why this proof is better than the previous line of argument. Following this introduction, the students should be ready for more 'Convincing and Proving' tasks, which appear at the end of this document.

#### *Providing guidance as students work on 'Convincing and Proving' tasks*

Whether your students work in groups or individually, many will ask for guidance while doing the tasks. The amount of guidance that students need should decline as they become familiar with this type of problem. Early in class, you are likely to provide guidance by being a critical audience for their explanations and arguments. Later, if your primary goal is to encourage students to struggle with solving the problems on their own (and learn that they can be effective mathematical thinkers), you may choose to provide very little assistance.

#### *Reporting out of individual or group work*

If you decide to come together as a large group to discuss what students came up with (or report out), it is again helpful to decide the degree to which you will participate in these discussions, which will depend upon your goals for the session. For instance, you can facilitate the students' discussion, having them defend their ideas and write their ideas on the board, while adding almost none of your own. This discussion is the whole essence of convincing and proving; in fact, your role in this situation could be to provoke critical discussions of different solutions. At some point, you might want to review the session, focusing on the quality of different explanations, some techniques for testing mathematical conjectures, the characteristics of different sorts of 'non-proofs', or identifying different sorts of proof.

#### *Formal and informal use of 'Convincing and Proving' tasks*

These tasks can be used formally or informally. In formal assessment (where you grade the assignment as an examination), do not intervene except where specified. Even modest interventions – reinterpreting instructions, suggesting ways to begin, offering prompts when students appear to be stuck – have the potential to alter the task for the student significantly. In informal assessment (an exercise, graded or non-graded), you may want to be less rigid in giving the students help. Under these circumstances, you may reasonably decide to do some coaching, talk with students as they work on the task, or pose questions when they seem to get stuck. In these instances you may be using the tasks for informal assessments—observing what strategies students favor, what kinds of questions they ask, what they seem to understand and what they are struggling with, and what kinds of prompts get them unstuck. This can be extremely useful information in helping you make ongoing instructional and assessment decisions. However, as students have more experiences with these kinds of tasks, the amount of coaching you do should decline and students should rely less on this kind of assistance.

#### *Group work versus individual work*

The communicative nature of "Convincing and Proving" tasks makes for great group discussions. Students can compare opinions and discuss the qualities of their various explanations. This may help them to refine their arguments. The CL1 Collaborative Learning site can provide instructions on how to use group work effectively within the classroom. However, individual work may give you more clues as to each student's sophistication with this type of problem.

#### *Presumed background knowledge*

Most "Convincing and Proving" tasks do require some background in algebra, which provides students a way to expressing generality. Many of the explanations may be attempted initially without recourse to algebra, since examples and counterexamples are sought. Eventually, however, the student will want to know if the result always works, and in such cases the need for algebra becomes apparent to the student. Some basic knowledge of geometry will also be needed for some problems.

### Step-by-Step instructions

1. Prepare by reading through the "Convincing and Proving" tasks on your own and coming up with your own solutions.
2. Hand out copies of the task to students, either working individually or in groups. You may like to use the introduction above under "Suggestions for use."
3. State the your goals for the "Convincing and Proving" task, emphasizing that students should try to find examples or counter examples, and write convincing explanations or reasons as to why they believe a statement to be always, sometimes, or never true. For the second set of tasks, they will need to identify flaws in someone else's reasoning.
4. Walk around and listen to students as they discuss and work through the problems, providing guidance as necessary.
5. Have students present their solutions, either in written or verbal form.

### Variations

The tasks included in this site can be downloaded and used without modification. If you choose to develop your own "Convincing and Proving" task, you can follow the pattern used in these tools.

For the first type of task, where mathematical statements are to be categorized as "always, sometimes or never true," it is often helpful to choose statements which are in fact common misconceptions that are often regarded as "always true" when they are in fact only true over a limited domain. Thus, you could choose statements such as, "When you add two fractions, you simply add the tops and the bottoms" or, "If you double the circumference of a circle you double its area." Note that while it might be quite easy to quickly dismiss these as not being "always true," finding all the cases when they are "sometimes true" is demanding. Thus, all students, regardless of background and ability level, can be challenged.

Designing the second type of task is more difficult. These require students to evaluate given "proofs" and discover flaws in them. Designing plausible errors can be quite tricky. One possibility is to set the students a normal "proof task" and select some of their own responses to analyze. These responses could be typed, copied (with names removed) and circulated to the whole class. Everyone can then discuss and assess their worth. As it is very uncomfortable to have your ideas pulled apart by your peers, anonymity should be carefully preserved. There are many famous examples of fallacious proofs which students may enjoy looking at in books, such as Eugene Northrop's 'Riddles in Mathematics' (Pelican).

One sample 'proof' that  $1 = 2$  is shown below:

Let	$x = y$
Then	$xy = y^2$
and	$xy - x^2 = y^2 - x^2$
and	$x(y - x) = (y + x)(y - x)$
Dividing by	$(y - x)$
Yields	$x = y + x$
But	$x = y$

So	$x = 2x$
Dividing by	$x$
Yields	$1 = 2$

### Analysis

In the first type of task, students' work can be measured against three criteria:

- whether students can understand a statement and test it in particular cases;
- whether the student can identify suitable examples or counterexamples which will suggest the truth or falsehood of a given statement; and,
- whether the student can provide a convincing argument or 'proof' to support their conclusion.

In the second type of task, the students work may be measured against two criteria:

- whether they are able to identify a correct proof and explain their choice; and,
- whether they can find mistakes or errors in logic within other given 'proofs'.

This generic scoring rubric may be modified and adapted for specific tasks.

**Figure 1: Scoring rubric for "Always, sometimes, or never true" tasks**

Category of performance	Typical response
The student needs significant instruction	<p>Student can understand a general statement, but cannot test it in specific cases.</p> <p>Student can only follow a small part of the given proof and cannot even begin to evaluate it.</p>
The student needs some instruction	<p>Student can understand a general statement, and can make up an example to test it. This example may not be well chosen and may lead the student to the wrong conclusion about the statement.</p> <p>Student can follow a given proof. The student cannot begin to explain, however, <i>why</i> or <i>where</i> it is flawed.</p>
The student's work needs to be revised	<p>Student can understand a general statement, and can test it with a suitable range of examples. The student draws an appropriate conclusion, but does not attempt to prove or justify it in general.</p> <p>Student can follow a given proof, and can correctly see that it is flawed. The student makes a partial attempt to explain the nature of the error, but this explanation is unclear or has significant omissions.</p>
The student's work meets the essential demands of the task	<p>Student can understand a general statement, and can test it with a suitable range of examples. The student draws an appropriate conclusion, and</p>

	<p>makes a good attempt to prove or justify it in general.</p> <p>Student can follow a given proof, and can correctly see that it is flawed. The student makes a good attempt to explain the nature of the error.</p>
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**Malcolm Swan**  
 Mathematics Education  
 University of Nottingham  
 Malcolm.Swan@nottingham.ac.uk



Most assessment practices seem to emphasise the reproduction of imitative, standardised techniques. I want something different for my students. I want them to become mathematicians - not rehearse and reproduce bits of mathematics.

I use the five 'mathematical thinking' tasks to stimulate discussion between students. They share solutions, argue in more logical, reasoned ways and begin to see mathematics as a powerful, creative subject to which they can contribute. Its much more fun to try to think and reach solutions collaboratively. Assessment doesn't have to be an isolated, threatening business.

**Not just answers, but approaches.**

Malcolm Swan is a lecturer in Mathematics Education at University of Nottingham and is a leading designer on the MARS team. His research interests lie in the design of teaching and assessment. He has worked for many years on research and development projects concerning diagnostic teaching (including ways of using misconceptions to promote long term learning), reflection and metacognition and the assessment of problem solving. For five years he was Chief Examiner for one of the largest examination boards in England. He is also interested in teacher development and has produced many courses and resources for the inservice training of teachers.

**Jim Ridgway**  
 School of Education  
 University of Durham  
 Jim.Ridgway@durham.ac.uk



*Thinking mathematically is about developing habits of mind that are always there when you need them - not in a book you can look up later.*

For me, a big part of education is about helping students develop uncommon common sense. I want students to develop ways of thinking that cross boundaries - between courses, and between mathematics and daily life.

People should be able to tackle new problems with some confidence - not with a sinking feeling 'we didn't do that yet'. I wanted to share a range of big ideas concerned with understanding complex situations, reasoning from evidence, and judging the likely success of possible solutions before



they were tried out. One problem I had is that my students seemed to learn things in 'boxes' that were only opened at exam time. Thinking mathematically is about developing habits of mind that are always there when you need them - not in a book you can look up later.

You can tell the teaching is working when mathematical thinking becomes part of everyday thinking. Sometimes it is evidence that the ideas have become part of the mental toolkit used in class - 'lets do a Fermi [make a plausible estimate] on it'. Sometimes it comes out as an anecdote. On graduate told me a story of how my course got him into trouble. He was talking with a senior clinician about the incidence of a problem in child development, and the need to employ more psychologists to address it. He 'did a Fermi' on the number of cases (wildly overestimated) and the resource implications (impossible in the circumstances). He said there was a silence in the group...you just don't teach the boss how to suck eggs, even when he isn't very good at it. He laughed.

Jim Ridgway is Professor of Education at the University of Durham, and leads the MARS team there. Jim's background is in applied cognitive psychology. As well as kindergarten to college level one assessment, his interests include the uses of computers in schools, fostering and testing higher order skills, and the study of change. His work on assessment is diverse, and includes, the selection of fast jet pilots, and cognitive analyses of the processes of task design. In MARS hhe has special responsibility for data analysis and psychometric issues, and for the CL-1 work.

### **About MARS**

The Mathematics Assessment Resource Service, MARS, offers a range of services and materials in support of the implementation of balanced performance assessment in mathematics across the age range K to CL-1. MARS is funded by the US National Science Foundation, and builds on earlier funding which began in 1992 for the Balanced Assessment Project (BA) from which MARS grew.

MARS offers effective support in:

*The Design of Assessment Systems:* assessment systems are tailored to the needs of specific clients. Design ranges from the contribution of individual tasks, through to full scale collaborative work on test development, scoring and reporting. Clients include Cities, States, and groups concerned with educational effectiveness, such as curriculum projects and professional development initiatives.

*Professional Development for Teachers:* most teachers need help in preparing their students for the much wider range of task types that balanced performance assessment involves. MARS offers professional development workshops for district leadership and 'mentor teachers', built on materials that are effective when used later by such leaders with their colleagues in school.

*Developing Design Skills:* many clients have good reasons to develop their own assessment, either for individual student assessment or for system monitoring. Doing this well is a challenge. MARS works with design teams in both design consultancy and the further development of the team's own design skills.

To support its design team, MARS has developed a database, now with around 1000 interesting tasks across the age range, on which designers can draw, modify or build, to fit any particular design challenge.